Math 1552

Sections 10.1: Sequences

Math 1552 lecture slides adapted from the course materials
By Klara Grodzinsky (GA Tech, *School of Mathematics*, Summer 2021)

Today's Learning Goals

- Use proper notation to denote a sequence.
- Understand how to find lower and upper bounds for sequences.
- Determine if a sequence is monotonic.
- Find limits of sequences when possible.

Sequences

A *sequence* is a *function* from the set of positive integers to the set of real numbers.

$$\{a_n\} = \{f(n)\} = a_1, a_2, \ldots, a_k, \ldots$$

a_n is called the n^{th} term

OR

$$\{a_n\}_{n\geq 1} = \{a_1, a_2, a_3, \ldots\}$$

$$\{f(n)\}_{n=0}^{\infty} = \{f(0), f(1), f(2), \ldots\}$$

The values of *n* are all positive integers, unless otherwise specified, e.g.,

Example:

Find an expression for the general term of the sequence below:

$$-\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \frac{5}{6}, \ldots$$

$$\text{A)}\ a_n = \frac{(-1)^n n}{n+1}$$

$$\text{B)}\ a_n = \frac{(-1)^{n+1} n}{n+1}$$

$$\text{C)}\ a_n = \frac{(-1)^n (n+1)}{n+2}$$

$$\text{D)}\ a_n = \frac{(-1)^{n+1} (n+1)}{n+2}$$

LUB and GLB

- An *upper bound* of a set S is a number *M* that is greater than or equal to each element in S.
- The smallest possible upper bound is called the *least upper bound* (*l.u.b.*) – *cf. the supremum*.

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- A *lower bound* of a set S is a number *m* that is less than or equal to each element in S.
- The largest possible lower bound is called the *greatest lower bound* (*g.l.b.) – cf. the infimum.*

Example:

Find the l.u.b. and g.l.b. of the sequence: $\left\{\frac{n+1}{n}\right\}$

A. l.u.b.=1, g.l.b.=
B. l.u.b.=2, g.l.b.=
C. l.u.b.=2, g.l.b.=
D. No l.u.b., g.l.b.=

Monotone Sequences

A sequence is called ***monotonic*** if any one of the following statements holds:

$(i)\ a_n < a_{n+1}$ for all n (strictly increasing) *(cf. non-decreasing)*

$(ii)\ a_n \leq a_{n+1}$ for all n (monotonically increasing) *(cf. non-decreasing)*

$(iii)\ a_n > a_{n+1}$ for all n (strictly decreasing) *(cf. non-increasing)*

$(iv)\ a_n \geq a_{n+1}$ for all n (monotonically decreasing) *(cf. non-increasing)*

Limit of a Sequence

Let $\{a_n\}$ be a sequence. If $\lim_{n \to \infty} a_n = L$,

then L is the ***limit*** of this sequence

If the sequence has a finite limit L, then the sequence is said to ***converge*** to L.

Otherwise, the sequence is said to ***diverge***.

Convergence Theorem

If a sequence $\{a_n\}_{n\geq 0}$ is ***monotonic*** and ***bounded***, then it converges (*to some finite limit L*).

If the sequence is *increasing*, then *L*=l.u.b.

If the sequence is *decreasing*, then *L*=g.l.b.

Equivalent statement:

An unbounded sequence diverges.

Example A:

Determine whether the sequence converges. If so, find the limit.

$$\left\{ \frac{n^2}{n+1} \right\}_{n \geq 1}$$

Example B:

Determine whether the sequence converges. If so, find the limit. $\{(-3)^n\}_{n\geq 1}$

Example C:

Determine whether the sequence converges. If so, find the limit.

$$\left\{ \frac{(-1)^n}{2^n} \right\}_{n \geq 1}$$

Example D:

Determine whether the sequence converges. If so, find the limit. $\left\{\frac{2^n}{n!}\right\}_{n\geq 1}$

Example E:

Determine whether the sequence converges. If so, find the limit.

$$\left\{\sin\left(\frac{n\pi}{2}\right)\right\}_{n\geq 1}$$

Example: Find the limit of the following sequence, if it exists: $\left\{\dfrac{2n+1}{1-3n}\right\}$ *(Justify your answer carefully.)*

A. 0
B. -2/3
C. 2/3
D. Diverges

Some Common Limits (memorize)

1) If $x > 0$, then $\lim_{n\to\infty} x^{1/n} = 1$.

2) If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

3) If $\alpha > 0$, then $\lim_{n\to\infty} \frac{1}{n^\alpha} = 0$.

4) $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

5) $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$

6) $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$

7) $\lim_{n\to\infty} n^{1/n} = 1$

8) If p is a positive integer, then:

$$\lim_{n\to\infty} \frac{a_p n^p + \cdots + a_1 n + a_0}{b_p n^p + \cdots + b_1 n + b_0} = \frac{a_p}{b_p}$$

(Do you see why?)

An interesting example

Why is the harmonic series divergent?
Can we prove that it diverges using the material we have seen so far?

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Consider using a Riemann sum to approximate the sums

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

for integers n where we let n go to infinity where we take the side widths of the rectangles on the x-axis to be one.

Recall: $\ln(x) = \int_1^x \frac{dt}{t}$

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Sources for figures:
https://www.cantorsparadise.com/the-euler-mascheroni-constant-4bd34203aa01
https://brilliant.org/wiki/euler-mascheroni-constant/

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Recall: $\ln(x) = \int_1^x \frac{dt}{t}$

Define: $T_n = H_n - \ln n$

We can show using <u>elementary methods</u> that

$$0 < \frac{1}{n} < T_n < 1, \text{ for all } n \geq 1$$

AND $T_{n+1} < T_n, \text{ for all } n \geq 1$

$$\implies \gamma = \lim_{n\to\infty} T_n$$

EXISTS!
(by monotonicity and boundedness)

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This constant is called *Euler-Mascheroni's gamma*
(or the *Euler gamma* constant for short):

$$\gamma = \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \ln(n) \right]$$

$$= \int_1^{\infty} \left(\frac{1}{\lfloor x \rfloor} - \frac{1}{x} \right) dx$$

$$\approx 0.5772156649015328606065120$$

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$$\gamma = \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right] = \int_1^\infty \left(\frac{1}{\lfloor x\rfloor} - \frac{1}{x}\right)dx \approx 0.5772156649015328606065120$$

The *harmonic series* is an example of a
p-series (with p=1) that diverges.
Now you can see why!

Challenge problem on limits of sequences I:

Suppose that $a_n = \begin{cases} 1 + \frac{1}{a_{n-1}}, & \text{if } n \geq 1; \\ 1, & \text{if } n = 0. \end{cases}$

Does the sequence converge? If so, what is $\lim_{n\to\infty} a_n$?

Challenge problem on limits of sequences II:

Suppose that $b_n = \begin{cases} b_{n-1} + 2b_{n-2}, & \text{if } n \geq 2; \\ 1, & \text{if } n = 1; \\ 2, & \text{if } n = 0. \end{cases}$

Does the sequence converge? If so, what is $\lim_{n\to\infty} b_n$?

Bonus problems on limits I:

Evaluate the following limit:

$$\lim_{n\to\infty}\left(1-\frac{x}{n}+\frac{x^2}{n^2}\right)^n$$

Bonus problems on limits II:

Evaluate the following limit:

$$\lim_{n\to\infty}\left(1+\frac{x^2}{n^2}\right)^{n^2}$$

Bonus problems on limits III (extra):

Show that $$\lim_{\alpha \to 0^+} \left(\frac{1 - e^{-\alpha v}}{\alpha} \right)^x = v^x, \alpha > 0, v > 0$$

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Sections 10.1: Review of Sequences

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Review Question: Which of the following sequences converge?

(A) $\left\{\dfrac{2n+1}{1-3n}\right\}$

(B) $\left\{(-1)^n\right\}$

(C) $\left\{\dfrac{2^n}{n!}\right\}$

(D) $\left\{\left(1+\dfrac{4}{n}\right)^n\right\}$

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Sections 10.2: Infinite Series

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Learning Goals

- Understand what is meant by an infinite series
- Understand the general rule of when an infinite series converges
- Identify geometric series and find their sums
- Identify telescoping series and find their sums
- Determine convergence or divergence with the nth term test

Recall: Limit of a Sequence

Let $\{a_n\}$ be a sequence. If $\lim_{n\to\infty} a_n = L$,

then L is the ***limit*** of this sequence

If the sequence has a finite limit *L*, then the sequence is said to ***converge*** to L.

Otherwise, the sequence ***diverges***.

Review of Sigma Notation

Recall from the sections on Riemann sums that

$$\sum_{k=0}^{n} a_k = a_0 + a_1 + a_2 + \ldots + a_n$$

$$\sum_{k=1}^{n} 1 = n, \text{ so } \sum_{k=0}^{n} 1 = n+1$$

$$\sum_{k=m}^{n} (a_k + b_k) = \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k$$

$$\sum_{k=m}^{n} ca_k = c\sum_{k=m}^{n} a_k$$

(Linearity)

$$\sum_{k=0}^{m} a_k + \sum_{k=m+1}^{n} a_k = \sum_{k=0}^{n} a_k$$

(Linearity)

Infinite Series

An *infinite series* is a *sum* of infinitely many terms:

$$\sum_{k=0}^{\infty} a_k = a_o + a_1 + a_2 + a_3 + \ldots + a_n + \ldots.$$

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$$\sum_{k=0}^{\infty} a_k = a_o + a_1 + a_2 + a_3 + \ldots + a_n + \ldots.$$

The series *converges* if the sequence of partial sums converges.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=0}^{n} a_k = L$$

The series *diverges* otherwise.

Which of these series do you think converges?

(That is, à priori – we will cover precise criteria for each case in the next slides.)

(A) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

(B) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)}$

(C) $\displaystyle\sum_{\mathrm{n}=0}^{\infty} \left(\frac{2}{3}\right)^{n}$

(D) None of these

The Harmonic Series

The ***Harmonic Series*** ***<u>diverges</u>*****!**

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

(Recall that we saw a proof of this fact in the Week 5 slides!)

Telescoping Series

- A telescoping series has the form:

$$\sum_{n=1}^{\infty} \frac{1}{(n+a)(n+b)}$$

- These series ***converge***.
- To find the sum, use *partial fractions*.

An Example:

Evaluate the following sum:

$$S = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} 2^k (3k+1)}{(k+1)(k+3)}$$

Geometric Series

- A geometric series has the form:

$$\sum_{n=0}^{\infty} r^n$$

- It ***converges* when $|r|<1$** and ***diverges*** otherwise.
- If $|r|<1$, the sum is:

$$\frac{1}{1-r}.$$

Example 1.1:

Sum the series: $\sum_{n=2}^{\infty} \frac{5^{n-1} + 3 \cdot 2^{3n}}{9^n}$

Example 1.2:

Use series to write the decimal 1.42424242... as a *rational number*.

Divergence (n^{th} term) Test

Given $\sum_{n=0}^{\infty} a_n$, first find $\lim_{n \to \infty} a_n$.

If $\lim_{n \to \infty} a_n \neq 0$, then the series **DIVERGES**

Otherwise, the test is *INCONCLUSIVE*

and you must try another test.

Important: ***n^{th} term test only tests for divergence!!***

- If the limit of the terms is equal to 0, you do not have enough information!
- For instance:
  - The harmonic series, the terms go to 0 but the series diverges!
  - Telescoping series, the terms go to 0 and these series converge!
- So… in order to converge, we need the limit to go to zero, but it is NOT a sufficient condition to determine convergence!

Example A:

Does the series diverge by the n^{th} term test?

$$\sum_{k=1}^{\infty}\left(1+\frac{3}{k}\right)^{k}$$

Example B:

Does the series diverge by the n^{th} term test?

$$\sum_{k=2}^{\infty} \frac{3k}{5k-7}$$

Example C:

Does the series diverge by the n^{th} term test?

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3+6}}$$

Which statement is always true?

$$\text{If } \lim_{n \to \infty} a_n = 0, \text{ then:}$$

A. The series converges.

B. The sequence converges.

C. The sequence of partial sums converges.

D. The series diverges.

Some Convergence Theorems

(1) If $\sum a_n$ and $\sum b_n$ both converge, then $\sum (a_n \pm b_n)$ also converges.

(2) If $\sum a_n$ converges, then $\sum ca_n$ also converges for any $c \in \Re$.

(3) If $\sum_{n=j}^{\infty} a_n$ converges, so does $\sum_{n=0}^{\infty} a_n$.

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Sections 10.3, 10.4 and 10.5 : Convergence Tests for Infinite Series

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Learning Goals

- Learn how to apply the integral, comparison, limit comparison, ratio and root series to determine whether an infinite series converges or diverges
- Learn when to apply which test
- Summarize the results into a formal mathematical justification

Quick review…

- The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ DIVERGES.

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- Telescoping series CONVERGE. Find the sum using partial fraction decompositions.

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- A geometric series

$$\sum_{k=0}^{\infty} r^k \qquad \textit{converges} \text{ to } \frac{1}{1-r} \text{ when } |r|<1$$

$$\textit{diverges} \text{ when } |r| \geq 1$$

Divergence (n^{th} term) Test

Given $\sum_{k=0}^{\infty} a_k$, first find $\lim_{n \to \infty} a_n$.

If $\lim_{n \to \infty} a_n \neq 0$, then the series **DIVERGES**!

Otherwise, the test is *INCONCLUSIVE*

and you must try another test.

Integral Test

Let f be a continuous, positive, and decreasing function. Then:

$$\sum_{k=1}^{\infty} f(k) \text{ converges if and only if } \int_1^{\infty} f(x)\,dx \text{ converges,}$$

$$\text{and diverges if and only if } \int_1^{N} f(x)\,dx \to \infty \text{ as } N \to \infty.$$

Example I:

Use the integral test to determine whether the series $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ converges:

Example II:

When does a p-series converge? $\sum_{k=1}^{\infty} \frac{1}{k^p}$ (p-series

Series we know:

- The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ DIVERGES.
- A geometric series

$$\sum_{k=0}^{\infty} r^k \qquad \textit{converges to } \frac{1}{1-r} \text{ when } |r|<1$$

$$\textit{diverges} \text{ when } |r| \geq 1$$

- A p-series

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \qquad \textit{converges} \text{ when } p>1$$

$$\textit{diverges} \text{ when } p \leq 1$$

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